

On the Reeh-Schlieder Property in Curved Spacetime

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*Dedicated to Klaas Landsman,
out of gratitude for the support he offered when it was most needed.*

Abstract

We attempt to prove the existence of Reeh-Schlieder states on curved spacetimes in the framework of locally covariant quantum field theory using the idea of spacetime deformation and assuming the existence of a Reeh-Schlieder state on a diffeomorphic (but not isometric) spacetime. We find that physically interesting states with a weak form of the Reeh-Schlieder property always exist and indicate their usefulness. Algebraic states satisfying the full Reeh-Schlieder property also exist, but are not guaranteed to be of physical interest.

1 Introduction

The Reeh-Schlieder theorem ([15]) is a result in axiomatic quantum field theory which states that for a scalar Wightman field in Minkowski spacetime any state in the Hilbert space can be approximated arbitrarily well by acting on the vacuum with operations performed in any prescribed open region. The physical meaning of this is that the vacuum state has very many non-local correlations and an experimenter in any given region can exploit the vacuum fluctuations by performing a suitable measurement in order to produce any desired state up to arbitrary accuracy.

In this paper we will investigate whether we can find states of a quantum field system in a curved spacetime which have the same property, (the Reeh-Schlieder property). We do this using the technique of spacetime deformation, as pioneered in [8] and as applied successfully to prove a spin-statistics theorem in curved spacetime in [18]. This means that we assume the existence of a Reeh-Schlieder state (i.e. a state with the Reeh-Schlieder property) in one spacetime and try to derive the existence of another state in a diffeomorphic (but not isometric) spacetime which also has the Reeh-Schlieder property. We will prove that for every given region there is a state in the physical state space that has the Reeh-Schlieder property for that particular region (but maybe not for all regions). Algebraic states with the full Reeh-Schlieder property also exist, i.e. states which have the Reeh-Schlieder property for all open regions simultaneously. However, their existence follows from an abstract existence principle and, consequently, such states are not guaranteed to be of any physical interest.

To keep the discussion as general as possible we will work in the axiomatic language known as locally covariant quantum field theory as introduced in [4] (see also [18], where some of these ideas already appeared, and [5] for a recent application). We outline this formulation in section 2 and our most important assumption there will be the time-slice axiom, which expresses the

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existence of a causal dynamical law. In section 3 we will prove the geometric results on spacetime deformation that we need and we will see what they mean for a locally covariant quantum field theory. Section 4 contains our main results on deforming one Reeh-Schlieder state into another one and it notes some immediate consequences regarding the type of local algebras and Tomita-Takesaki modular theory. As an example we discuss the free scalar field in section 5 and we end with a few conclusions.

2 Locally covariant quantum field theory

In this section we briefly describe the main ideas of locally covariant quantum field theory as introduced in [4]. It will also serve to fix our notation for the subsequent sections.

In the following any quantum physical system will be described by a C^* -algebra \mathcal{A} with a unit I , whose self-adjoint elements are the observables of the system. It will be advantageous to consider a whole class of possible systems rather than just one.

Definition 2.1 *The category \mathfrak{Alg} has as its objects all unital C^* -algebras \mathcal{A} and as its morphisms all injective $*$ -homomorphisms α such that $\alpha(I) = I$. The product of morphisms is given by the composition of maps and the identity map $\text{id}_{\mathcal{A}}$ on a given object serves as an identity morphism.*

A morphism $\alpha: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ expresses the fact that the system described by \mathcal{A}_1 is a sub-system of that described by \mathcal{A}_2 , which is called a super-system. The injectivity of the morphisms means that, as a matter of principle, any observable of a sub-system can always be measured, regardless of any practical restrictions that a super-system may impose.

A state of a system is represented by a normalised positive linear functional ω , i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$ and $\omega(I) = 1$. The set of all states on \mathcal{A} will be denoted by \mathcal{A}_1^{*+} . Not all of these states are of physical interest, so it will be convenient to have the following notion at our disposal.

Definition 2.2 *The category \mathfrak{States} has as its objects all subsets $S \subset \mathcal{A}_1^{*+}$, for all unital C^* -algebras \mathcal{A} in \mathfrak{Alg} , which are closed under convex linear combinations and under operations from \mathcal{A} (i.e. $\frac{\omega(A^*A)}{\omega(A^*A)} \in S$ if $\omega \in S$ and $A \in \mathcal{A}$ such that $\omega(A^*A) \neq 0$) and as its morphisms all maps $\alpha^*: S_1 \rightarrow S_2$ for which $S_i \subset (\mathcal{A}_i)_1^{*+}$, $i = 1, 2$, and α^* is the restriction of the dual of a morphism $\alpha: \mathcal{A}_2 \rightarrow \mathcal{A}_1$ in \mathfrak{Alg} , i.e. $\alpha^*(\omega) = \omega \circ \alpha$ for all $\omega \in S_1$. Again the product of morphisms is given by the composition of maps and the identity map id_S on a given object serves as an identity morphism.*

After these operational aspects we now turn to the physical ones. The systems we will consider are intended to model quantum fields living in a (region of) spacetime which is endowed with a fixed Lorentzian metric (a background gravitational field). The relation between sub-systems will come about naturally by considering sub-regions of spacetime. More precisely we consider the following:

Definition 2.3 *By the term globally hyperbolic spacetime we will mean a connected, Hausdorff, paracompact, C^∞ Lorentzian manifold $M = (\mathcal{M}, g)$ of dimension $d = 4$, which is oriented, time-oriented and globally hyperbolic.*

A subset $O \subset \mathcal{M}$ of a globally hyperbolic spacetime M is called causally convex iff for all $x, y \in O$ all causal curves from x to y lie entirely in O . A non-empty open set which is connected and causally convex is called a causally convex region or cc-region. A cc-region whose closure is compact is called a bounded cc-region.

The category \mathfrak{Man} has as its objects all globally hyperbolic spacetimes $M = (\mathcal{M}, g)$ and its morphisms Ψ are given by all maps $\psi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ which are smooth isometric embeddings (i.e. $\psi: \mathcal{M}_1 \rightarrow \psi(\mathcal{M}_1)$ is a diffeomorphism and $\psi_*g_1 = g_2|_{\psi(\mathcal{M}_1)}$) such that the orientation and time-orientation are preserved and $\psi(\mathcal{M}_1)$ is causally convex. Again the product of morphisms is given by the composition of maps and the identity map id_M on a given object serves as a unit.

A region O in a globally hyperbolic spacetime is causally convex if and only if O itself is globally hyperbolic (see [10] section 6.6), so a cc-region is exactly a connected globally hyperbolic region.

The image of a morphism is by definition a cc-region. Notice that the converse also holds. If $O \subset \mathcal{M}$ is a cc-region then $(O, g|_O)$ defines a globally hyperbolic spacetime in its own right. In this case there is a canonical morphism $I_{M,O} : O \rightarrow M$ given by the canonical embedding $\iota : O \rightarrow \mathcal{M}$. We will often drop $I_{M,O}$ and ι from the notation and simply write $O \subset M$.

The importance of causally convex sets is that for any morphism Ψ the causality structure of M_1 coincides with that of $\Psi(M_1)$ in M_2 :

$$\psi(J_{M_1}^\pm(x)) = J_{M_2}^\pm(\psi(x)) \cap \psi(\mathcal{M}_1), \quad x \in \mathcal{M}_1. \quad (1)$$

If this were not the case then the behaviour of a quantum physical system living in \mathcal{M}_1 could depend in an essential way on the super-system, which makes it practically impossible to study the smaller system as a sub-system in its own right. This possibility is therefore excluded from in mathematical framework.

Equation (1) allows us to drop the subscript in J_M^\pm if we introduce the convention that J^\pm is always taken in the largest spacetime under consideration. This simplifies the notation without causing any confusion, even when $O \subset M_1 \subset M_2$ with canonical embeddings, because then we just have $J^\pm(O) := J_{M_2}^\pm(O)$ and $J_{M_1}^\pm(O) = J^\pm(O) \cap \mathcal{M}_1$. Similarly we take by convention

$$\begin{aligned} D(O) &:= D_{M_2}(O), \\ O^\perp &:= O^{\perp_{M_2}} := \mathcal{M}_2 \setminus \overline{J(O)}, \end{aligned}$$

and we deduce from causal convexity that $D_{M_1}(O) = D(O) \cap \mathcal{M}_1$ and $O^{\perp_{M_1}} = O^\perp \cap \mathcal{M}_1$.

The following lemma gives some ways of obtaining causally convex sets in a globally hyperbolic spacetime.

Lemma 2.4 *Let $M = (\mathcal{M}, g)$ be a globally hyperbolic spacetime, $O \subset \mathcal{M}$ an open subset and $A \subset \mathcal{M}$ an achronal set. Then:*

1. *the intersection of two causally convex sets is causally convex,*
2. *for any subset $S \subset M$ the sets $I^\pm(S)$ are causally convex,*
3. *O^\perp is causally convex,*
4. *O is causally convex iff $O = J^+(O) \cap J^-(O)$,*
5. *$\text{int}(D(A))$ and $\text{int}(D^\pm(A))$ are causally convex,*
6. *if O is a cc-region, then $D(O)$ is a cc-region,*
7. *if $S \subset M$ is an acausal continuous hypersurface then $D(S)$, $D(S) \cap I^+(S)$ and $D(S) \cap I^-(S)$ are open and causally convex.*

Proof. The first two items follow directly from the definitions. The fourth follows from $J^+(O) \cap J^-(O) = \cup_{p,q \in O} (J^+(p) \cap J^-(q))$, which is contained in O if and only if O is causally convex. The fifth item follows from the first two and theorem 14.38 and lemma 14.6 in [13].

To prove the third item, assume that γ is a causal curve between points in O^\perp and $p \in \overline{J(O)}$ lies on γ . By perturbing one of the endpoints of γ in O^\perp we may ensure that the curve is time-like. Then we may perturb p on γ so that $p \in \text{int}(J(O))$ and γ is still causal. This gives a contradiction, because there then exists a causal curve from O through p to either x or y .

For the sixth statement we let $S \subset O$ be a smooth Cauchy surface for O (see [2]) and note that $D(O)$ is non-empty, connected and $D(O) = D(S)$. The causal convexity of O implies that $S \subset \mathcal{M}$ is acausal, which reduces this case to statement seven. The first part of statement seven is just lemma 14.43 and theorem 14.38 in [13]. The rest of statement seven follows from statement one and two together with the openness of $I^\pm(S)$. \square

We now come to the main set of definitions, which combine the notions introduced above (see [4]).

Definition 2.5 A locally covariant quantum field theory is a covariant functor $\mathbf{A} : \mathfrak{Man} \rightarrow \mathfrak{Alg}$, written as $M \mapsto \mathcal{A}_M$, $\Psi \mapsto \alpha_\Psi$.

A state space for a locally covariant quantum field theory \mathbf{A} is a contravariant functor $\mathbf{S} : \mathfrak{Man} \rightarrow \mathfrak{States}$, such that for all objects M we have $M \mapsto S_M \subset (\mathcal{A}_M)_1^{*+}$ and for all morphisms $\Psi : M_1 \rightarrow M_2$ we have $\Psi \mapsto \alpha_\Psi^*|_{S_{M_2}}$. The set S_M is called the state space for M .

When it is clear that $\Psi = I_{M,O}$ for a canonical embedding $\iota : O \rightarrow \mathcal{M}$ of a cc-region O in a globally hyperbolic spacetime \mathcal{M} , i.e. when $O \subset M$, we will often simply write $\mathcal{A}_O \subset \mathcal{A}_M$ instead of using $\alpha_{I_{M,O}}$. For a morphism $\Psi : M \rightarrow M'$ which restricts to a morphism $\Psi|_O : O \rightarrow O' \subset M$ we then have

$$\alpha_{\Psi|_O} = \alpha_\Psi|_{\mathcal{A}_O} \quad (2)$$

rather than $\alpha_{I_{M',O'}} \circ \alpha_{\Psi|_O} = \alpha_\Psi \circ \alpha_{I_{M,O}}$, as one can see from a commutative diagram.

The framework of locally covariant quantum field theory is a generalisation of algebraic quantum field theory (see [4, 9]). We now proceed to discuss several physically desirable properties that such a locally covariant quantum field theory and its state space may have (cf. [4], but note that our time-slice axiom is stronger).

Definition 2.6 A locally covariant quantum field theory \mathbf{A} is called causal iff for any two morphisms $\Psi_i : M_i \rightarrow M$, $i = 1, 2$ such that $\psi_1(\mathcal{M}_1) \subset (\psi_2(\mathcal{M}_2))^\perp$ in \mathcal{M} we have $[\alpha_{\Psi_1}(\mathcal{A}_{M_1}), \alpha_{\Psi_2}(\mathcal{A}_{M_2})] = \{0\}$ in \mathcal{A}_M .

A locally covariant quantum field theory \mathbf{A} with state space \mathbf{S} satisfies the time-slice axiom iff for all morphisms $\Psi : M_1 \rightarrow M_2$ such that $\psi(\mathcal{M}_1)$ contains a Cauchy surface for \mathcal{M}_2 we have $\alpha_\Psi(\mathcal{A}_{M_1}) = \mathcal{A}_{M_2}$ and $\alpha_\Psi^*(S_{M_2}) = S_{M_1}$.

A state space \mathbf{S} for a locally covariant quantum field theory \mathbf{A} is called locally quasi-equivalent iff for every morphism $\Psi : M_1 \rightarrow M_2$ such that $\psi(\mathcal{M}_1) \subset \mathcal{M}_2$ is bounded and for every pair of states $\omega, \omega' \in S_{M_2}$ the GNS-representations $\pi_\omega, \pi_{\omega'}$ of \mathcal{A}_{M_2} are quasi-equivalent on $\alpha_\Psi(\mathcal{A}_{M_1})$. The local von Neumann algebras $\mathcal{R}_{M_1}^\omega := \pi_\omega(\alpha_\Psi(\mathcal{A}_{M_1}))''$ are then *-isomorphic for all $\omega \in S_{M_2}$.

A locally covariant quantum field theory \mathbf{A} with a state space functor \mathbf{S} is called nowhere classical iff for every morphism $\Psi : M_1 \rightarrow M_2$ and for every state $\omega \in S_{M_2}$ the local von Neumann algebra $\mathcal{R}_{M_1}^\omega$ is not commutative.

Note that the condition $\psi_1(\mathcal{M}_1) \subset (\psi_2(\mathcal{M}_2))^\perp$ is symmetric in $i = 1, 2$. The causality condition formulates how the quantum physical system interplays with the classical gravitational background field, whereas the time-slice axiom expresses the existence of a causal dynamical law. The condition of a locally quasi-equivalent state space is more technical in nature and means that all states of a system can be described in the same Hilbert space representation as long as we only consider operations in a small (i.e. bounded) cc-region of the spacetime.

The condition that $\psi(\mathcal{M}_1)$ contains a Cauchy surface for \mathcal{M}_2 is equivalent to $D(\psi(\mathcal{M}_1)) = \mathcal{M}_2$, because a Cauchy surface $S \subset \mathcal{M}_1$ maps to a Cauchy surface $\psi(S)$ for $D(\psi(\mathcal{M}_1))$. On the algebraic level this yields:

Lemma 2.7 For a locally covariant quantum field theory \mathbf{A} with a state space \mathbf{S} satisfying the time-slice axiom, an object $(\mathcal{M}, g) \in \mathfrak{Man}$ and a cc-region $O \subset \mathcal{M}$ we have $\mathcal{A}_O = \mathcal{A}_{D(O)}$ and $S_O = S_{D(O)}$. If O contains a Cauchy surface of \mathcal{M} we have $\mathcal{A}_O = \mathcal{A}_\mathcal{M}$ and $S_O = S_\mathcal{M}$.

Proof. Note that both $(O, g|_O)$ and $(D(O), g|_{D(O)})$ are objects of \mathfrak{Man} (by lemma 2.4) and that a Cauchy surface S for O is also a Cauchy surface for $D(O)$. (The causal convexity of O in \mathcal{M} prevents multiple intersections of S .) The first statement then reduces to the second. Leaving the canonical embedding implicit in the notation, the result immediately follows from the time-slice axiom. \square

Finally we define the Reeh-Schlieder property, which we will study in more detail in the subsequent sections.

Definition 2.8 Consider a locally covariant quantum field theory \mathbf{A} with a state space \mathbf{S} . A state $\omega \in S_M$ has the Reeh-Schlieder property for a cc-region $O \subset \mathcal{M}$ iff

$$\overline{\pi_\omega(\mathcal{A}_O)\Omega_\omega} = \mathcal{H}_\omega$$

where $(\pi_\omega, \Omega_\omega, \mathcal{H}_\omega)$ is the GNS-representation of \mathcal{A}_M in the state ω . We then say that ω is a Reeh-Schlieder state for O . We say that ω is a (full) Reeh-Schlieder state iff it is a Reeh-Schlieder state for all cc-regions in \mathcal{M} .

3 Spacetime deformation

The existence of Hadamard states of the free scalar field in certain curved spacetimes was proved in [8] by deforming Minkowski spacetime into another globally hyperbolic spacetime. Using a similar but slightly more technical spacetime deformation argument [18] proved a spin-statistics theorem for locally covariant quantum field theories with a spin structure, given that such a theorem holds in Minkowski spacetime. In the next section we will assume the existence of a Reeh-Schlieder state in one spacetime and try to deduce along similar lines the existence of such states on a deformed spacetime. As a geometric prerequisite we will state and prove in the present section a spacetime deformation result employing similar methods as the references mentioned above.

First we recall the spacetime deformation result due to [8]:

Proposition 3.1 Consider two globally hyperbolic spacetimes M_i , $i = 1, 2$, with spacelike Cauchy surfaces C_i both diffeomorphic to C . Then there exists a globally hyperbolic spacetime $M' = (\mathbb{R} \times C, g')$ with spacelike Cauchy surfaces C'_i , $i = 1, 2$, such that C'_i is isometrically diffeomorphic to C_i and an open neighbourhood of C'_i is isometrically diffeomorphic to an open neighbourhood of C_i .

The proof is omitted, because the stronger result proposition 3.3 will be proved later on. Note, however, the following interesting corollary (cf. [4] section 4):

Corollary 3.2 Two globally hyperbolic spacetimes M_i with diffeomorphic Cauchy surfaces are mapped to isomorphic C^* -algebras \mathcal{A}_{M_i} by any locally covariant quantum field theory \mathbf{A} satisfying the time-slice axiom (with some state space \mathbf{S}).

Proof. Consider two diffeomorphic globally hyperbolic spacetimes M_i , $i = 1, 2$, let M' be the deforming spacetime of proposition 3.1 and let $W_i \subset \mathcal{M}_i$ be open neighbourhoods of the Cauchy surfaces $C_i \subset \mathcal{M}_i$ which are isometrically diffeomorphic under ψ_i to the open neighbourhoods $W'_i \subset \mathcal{M}'$ of the Cauchy surfaces $C'_i \subset \mathcal{M}'$. We may take the W_i and W'_i to be cc-regions (as will be shown in proposition 3.3), so that the Ψ_i (determined by ψ_i) are isomorphisms in \mathfrak{Man} . It then follows from lemma 2.7 that

$$\begin{aligned} \mathcal{A}_{M_1} &= \mathcal{A}_{W_1} = \mathcal{A}_{\psi_1^{-1}(W'_1)} = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{W'_1}) = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{M'}) \\ &= \alpha_{\Psi_1}^{-1} \circ \alpha_{\Psi_2}(\mathcal{A}_{M_2}), \end{aligned}$$

where the α_{Ψ_i} are $*$ -isomorphisms. This proves the assertion. \square

At this point a warning seems in place. Whenever g_1, g_2 are two Lorentzian metrics on a manifold \mathcal{M} such that both $M_i := (\mathcal{M}, g_i)$ are objects in \mathfrak{Man} , corollary 3.2 gives a $*$ -isomorphism α between the algebras \mathcal{A}_{M_i} . If $O \subset \mathcal{M}$ is a cc-region for g_1 then α is a $*$ -isomorphism from $\mathcal{A}_{(O, g_1)}$ into \mathcal{A}_{M_2} . However, the image cannot always be identified with $\mathcal{A}_{(O, g_2)}$, because O need not be causally convex for g_2 , in which case the object is not defined.

We now formulate and prove our deformation result. The geometric situation is schematically depicted in figure 1.

Proposition 3.3 Consider two globally hyperbolic spacetimes M_i , $i = 1, 2$, with diffeomorphic Cauchy surfaces and a bounded cc-region $O_2 \subset \mathcal{M}_2$ with non-empty causal complement, $O_2^\perp \neq \emptyset$. Then there are a globally hyperbolic spacetime $M' = (\mathcal{M}', g')$, spacelike Cauchy surfaces $C_i \subset \mathcal{M}_i$ and $C'_1, C'_2 \in \mathcal{M}'$ and bounded cc-regions $U_2, V_2 \subset \mathcal{M}_2$ and $U_1, V_1 \subset \mathcal{M}_1$ such that the following hold:

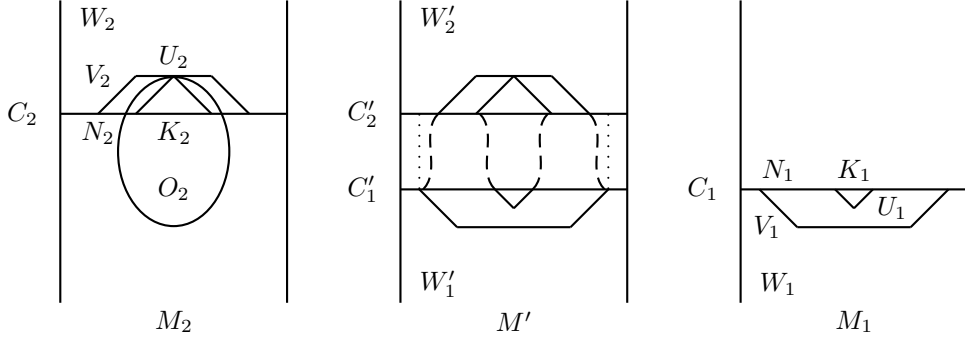


Figure 1: Sketch of the geometry of proposition 3.3.

- There are isometric diffeomorphisms $\psi_i : W_i \rightarrow W'_i$ where $W_1 := I^-(C_1)$, $W'_1 := I^-(C'_1)$, $W_2 := I^+(C_2)$ and $W'_2 := I^+(C'_2)$,
- $U_2, V_2 \subset W_2$, $U_2 \subset D(O_2)$, $O_2 \subset D(V_2)$,
- $U_1, V_1 \subset W_1$, $U_1 \neq \emptyset$, $V_1^\perp \neq \emptyset$, $\psi_1(U_1) \subset D(\psi_2(U_2))$ and $\psi_2(V_2) \subset D(\psi_1(V_1))$.

Proof. First we recall the result of [2] that for any globally hyperbolic spacetime (\mathcal{M}, g) there is a diffeomorphism $F : \mathcal{M} \rightarrow \mathbb{R} \times C$ for some smooth three dimensional manifold C in such a way that for each $t \in \mathbb{R}$ the surface $F^{-1}(\{t\} \times C)$ is a spacelike Cauchy surface. The pushed-forward metric $g' := F_*g$ makes $(\mathbb{R} \times C, g')$ a globally hyperbolic manifold, where g' is given by

$$g'_{\mu\nu} = \beta dt_\mu dt_\nu - h_{\mu\nu}. \quad (3)$$

Here dt is the differential of the canonical projection on the first coordinate $t : \mathbb{R} \times C \rightarrow \mathbb{R}$, which is a smooth time function; β is a strictly positive smooth function and $h_{\mu\nu}$ is a (space and time dependent) Riemannian metric on C . The orientation and time orientation of \mathcal{M} induce an orientation and time orientation on $\mathbb{R} \times C$ via F . (If necessary we may compose F with the time-reversal diffeomorphism $(t, x) \mapsto (-t, x)$ of $\mathbb{R} \times C$ to ensure that the function t increases in the positive time direction.) Applying the above to the M_i gives us two diffeomorphisms $F_i : \mathcal{M}_i \rightarrow \mathcal{M}'$, where $\mathcal{M}' = \mathbb{R} \times C$ as a manifold. Note that we can take the same C for both $i = 1, 2$ by the assumption of diffeomorphic Cauchy surfaces.

Define $O'_2 := F_2(O_2)$ and let t_{\min} and t_{\max} be the minimum and maximum value that the function t attains on the compact set $\overline{O'_2}$. We now prove that $F_2^{-1}((t_{\min}, t_{\max}) \times C) \cap O_2^\perp \neq \emptyset$. Indeed, if this were empty, then we see that $\overline{J(O_2)}$ contains $F_2^{-1}([t_{\min}, t_{\max}] \times C)$ and hence also $C_{\max} := F_2^{-1}(\{t_{\max}\} \times C)$ and $C_{\min} := F_2^{-1}(\{t_{\min}\} \times C)$. In fact, $C_{\min} \subset \overline{J^-(O_2)}$. Indeed, if $p := F_2^{-1}(t_{\min}, x)$ is in $\overline{J^+(O_2)}$ then we can consider a basis of neighbourhoods of p of the form $I^-(F_2^{-1}(t_{\min} + 1/n, x)) \cap I^+(F_2^{-1}(\{t_{\min} - 1/n\} \times C))$. If $q_n \in J^+(O_2)$ is in such a basic neighbourhood, then the same neighbourhood also contains a point $p_n \in O_2$. Hence, given a sequence q_n in $J^+(O_2)$ converging to p we find a sequence p_n in O_2 converging to p and we conclude that $p \in \overline{O_2} \subset \overline{J^-(O_2)}$. Similarly we can show that $C_{\max} \subset \overline{J^+(O_2)}$. It then follows that $I^+(C_{\max}) \subset \overline{J^+(O_2)}$ and $I^-(C_{\min}) \subset \overline{J^-(O_2)}$, so that $\overline{J(O_2)} = \mathcal{M}$ and $O_2^\perp = \emptyset$. This contradicts our assumption on O_2 , so we must have $F_2^{-1}((t_{\min}, t_{\max}) \times C) \cap O_2^\perp \neq \emptyset$. Then we may choose $t_2 \in (t_{\min}, t_{\max})$ such that $C_2 := F_2^{-1}(\{t_2\} \times C)$ satisfies $C_2 \cap O_2 \neq \emptyset$ and $C_2 \cap O_2^\perp \neq \emptyset$. We define $C'_2 := F_2(C_2)$, $W_2 := I^+(C_2)$ and $W'_2 := (t_2, \infty) \times C$.

Note that $C_2 \cap J(\overline{O_2})$ is compact. (We can $\overline{O_2}$ by a finite number of open sets of the form $I^\pm(q_n)$ and apply theorem 8.3.12 in [20] to each point q_n .) It follows that we can find relatively compact open sets $K, N \subset C$ such that $K'_2 := \{t_2\} \times K$, $K_2 := F_2^{-1}(K'_2)$, $N'_2 := \{t_2\} \times N$ and $N_2 := F_2^{-1}(N'_2)$ satisfy $K \neq \emptyset$, $\overline{N} \neq C$, $\overline{K_2} \subset O_2$ and $C_2 \cap J(\overline{O_2}) \subset N_2$. We let $C_{\max} := F_2^{-1}(\{t_{\max}\} \times C)$ and define $U_2 := D(K_2) \cap I^+(K_2) \cap I^-(C_{\max})$ and $V_2 := D(N_2) \cap I^+(N_2) \cap I^-(C_{\max})$. It follows from lemma

2.4 that U_2, V_2 are bounded cc-regions in M_2 . Clearly $U_2, V_2 \subset W_2$, $U_2 \subset D(O_2)$, $O_2 \subset D(V_2)$ and $V_2^\perp \neq \emptyset$.

Next we choose $t_1 \in (t_{\min}, t_2)$ and define $C'_1 := \{t_1\} \times C$, $C_1 := F_1^{-1}(C'_1)$, $W_1 := I^-(C_1)$ and $W'_1 := (-\infty, t_1) \times C$. Let $N', K' \subset C$ be relatively compact connected open sets such that $K' \neq \emptyset$, $\overline{N'} \neq C$, $\overline{K'} \subset K$ and $\overline{N} \subset N'$. We define $N'_1 := \{t_1\} \times N'$, $K'_1 := \{t_1\} \times K'$, $N_1 := F_1^{-1}(N'_1)$, $K_1 := F_1^{-1}(K'_1)$ and $C_{\min} := F_1^{-1}(\{t_{\min}\} \times C)$. Let $U_1 := D(K_1) \cap I^-(K_1) \cap I^+(C_{\min})$ and $V_1 := D(N_1) \cap I^-(N_1) \cap I^+(C_{\min})$. Again by lemma 2.4 these are bounded cc-regions in \mathcal{M}_1 . Note that $U_1, V_1 \subset W_1$ and $V_1^\perp \neq \emptyset$.

The metric g' of \mathcal{M}' is now chosen to be of the form

$$g'_{\mu\nu} := \beta dt_\mu dt_\nu - f \cdot (h_1)_{\mu\nu} - (1-f) \cdot (h_2)_{\mu\nu}$$

where we have written $((F_i)_* g_i)_{\mu\nu} = \beta_i dt_\mu dt_\nu - (h_i)_{\mu\nu}$, f is a smooth function on \mathcal{M}' which is identically 1 on W'_1 , identically 0 on W'_2 and $0 < f < 1$ on the intermediate region $(t_1, t_2) \times C$ and β is a positive smooth function which is identically β_i on W'_i . It is then clear that the maps F_i restrict to isometric diffeomorphisms $\psi_i: W_i \rightarrow W'_i$.

The function β may be chosen small enough on the region $(t_1, t_2) \times C$ to make (\mathcal{M}, g') globally hyperbolic. (As pointed out in [8] in their proof of proposition 3.1, choosing β small “closes up” the light cones and prevents causal curves from “running off to spatial infinity” in the intermediate region.) Furthermore, using the compactness of $(t_1, t_2) \times N'$ and the continuity of $(h_i)_{\mu\nu}$ we see that we may choose β small enough on this set to ensure that any causal curve through $\overline{K'_1}$ must also intersect $\overline{K'_2}$ and any causal curve through $\overline{N'_2}$ must also intersect N'_1 . This means that $\overline{K'_1} \subset D(K'_2)$ and $\overline{N'_2} \subset D(N'_1)$ and hence $\psi_1(U_1) \subset D(\psi_2(U_2))$ and $\psi_2(V_2) \subset D(\psi_1(V_1))$. This completes the proof. \square

The analogue of corollary 3.2 for the situation of proposition 3.3 is:

Proposition 3.4 *Consider a locally covariant quantum field theory \mathbf{A} with a state space \mathbf{S} satisfying the time-slice axiom and two globally hyperbolic spacetimes M_i , $i = 1, 2$ with diffeomorphic Cauchy surfaces. For any bounded cc-region $O_2 \subset \mathcal{M}_2$ with non-empty causal complement there are bounded cc-regions $U_1, V_1 \subset \mathcal{M}_1$ and a $*$ -isomorphism $\alpha: \mathcal{A}_{M_2} \rightarrow \mathcal{A}_{M_1}$ such that $V_1^\perp \neq \emptyset$ and*

$$\mathcal{A}_{U_1} \subset \alpha(\mathcal{A}_{O_2}) \subset \mathcal{A}_{V_1}. \quad (4)$$

Moreover, if the spacelike Cauchy surfaces of the M_i are non-compact and $P_2 \subset \mathcal{M}_2$ is any bounded cc-region, then there are bounded cc-regions $Q_2 \subset \mathcal{M}_2$ and $P_1, Q_1 \subset \mathcal{M}_1$ such that $Q_i \subset P_i^\perp$ for $i = 1, 2$ and

$$\alpha(\mathcal{A}_{P_2}) \subset \mathcal{A}_{P_1}, \quad \mathcal{A}_{Q_1} \subset \alpha(\mathcal{A}_{Q_2}), \quad (5)$$

where α is the same $*$ -isomorphism as in the first part of this proposition.

Proof. We apply proposition 3.3 to obtain sets U_i, V_i with and isomorphisms $\Psi_i: W_i \rightarrow W'_i$ associated to the isometric diffeomorphisms ψ_i . As in the proof of corollary 3.2 the Ψ_i give rise to $*$ -isomorphisms α_{Ψ_i} and $\alpha := \alpha_{\Psi_1}^{-1} \circ \alpha_{\Psi_2}$ is a $*$ -isomorphism from \mathcal{A}_{M_2} to \mathcal{A}_{M_1} . Using the properties of U_i, V_i stated in proposition 3.3 we deduce:

$$\begin{aligned} \mathcal{A}_{U_1} &= \alpha_{\Psi_1}^{-1}(\mathcal{A}_{U'_1}) \subset \alpha_{\Psi_1}^{-1}(\mathcal{A}_{D(U'_2)}) = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{U'_2}) = \alpha(\mathcal{A}_{U_2}) \subset \alpha(\mathcal{A}_{O_2}) \\ &\subset \alpha(\mathcal{A}_{V_2}) = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{V'_2}) \subset \alpha_{\Psi_1}^{-1}(\mathcal{A}_{D(V'_1)}) = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{V'_1}) = \mathcal{A}_{V_1}. \end{aligned}$$

Here we repeatedly used equation (2) and lemma 2.7 (the time slice axiom). This proves the first part of the proposition.

Now suppose that the Cauchy-surfaces are non-compact and let P_2 be any bounded cc-region. We refer to figure 2 for a depiction of this part of the proof.

First choose Cauchy surfaces $T_2, T_+ \subset W_2$ such that $T_+ \subset I^+(T_2)$. Note that $J(\overline{P_2}) \cap T_2$ is compact, so it has a relatively compact connected open neighbourhood $N_2 \subset T_2$. Choosing T_+ appropriately we see that $R := D(N_2) \cap I^+(N_2) \cap I^-(T_+)$ is a bounded cc-region in \mathcal{M}_2 by lemma 2.4 and as usual we set $R' := \psi_2(R)$.

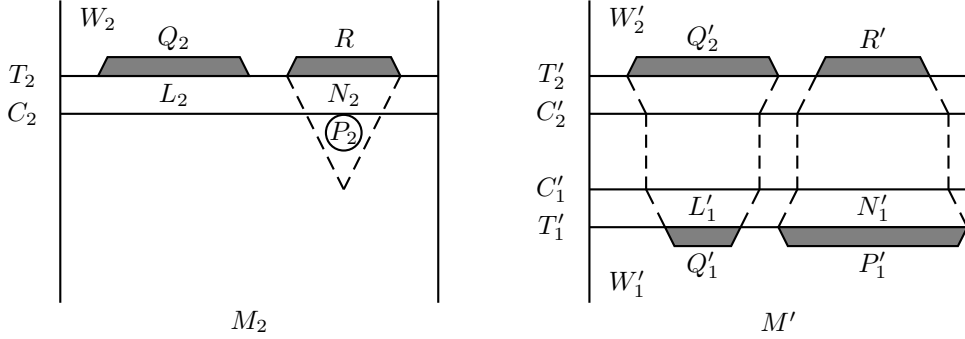


Figure 2: Sketch of the proof of the second part of proposition 3.4.

Now let $T'_-, T'_1 \subset W'_1$ be Cauchy surfaces such that $T'_- \subset I^-(T'_1)$ and note that $J(\overline{R'}) \cap T'_1$ is again compact, so we can find a relatively compact connected open neighbourhood $N'_1 \subset T'_1$ and use lemma 2.4 to define the bounded cc-region $P'_1 := D(N'_1) \cap I^-(N'_1) \cap I^+(T'_-)$ and $P_1 := \psi_1^{-1}(P'_1)$.

Now let $L'_1 \subset T'_1$ be a connected relatively compact set such that $L'_1 \cap N'_1 = \emptyset$. Such an L'_1 exists because T'_1 is non-compact. Define $Q'_1 := D(L'_1) \cap I^-(L'_1) \cap I^+(T'_-)$ and $Q_1 := \psi_1^{-1}(Q'_1)$. We see that $Q_1 \subset P_1^\perp$ is a bounded cc-region and $Q'_1 \subset D(\psi_2(L_2))$ where $L_2 \subset T_2 \setminus N$ is a relatively compact open set. In fact, we can choose L_2 to be connected because Q'_1 lies in a connected component C of $D(\psi_2(T_2 \setminus N))$. We now define the bounded cc-region $Q_2 := D(L_2) \cap I^+(L_2) \cap I^-(T_+)$ and $Q'_2 := \psi_2(Q_2)$, so that $Q_1 \subset P_1^\perp$ and $Q'_1 \subset D(Q'_2)$.

So far the geometry of the proof. Now note that $\mathcal{A}_{P_2} \subset \mathcal{A}_R$ by lemma 2.7 on $D(N_2) \cap I^+(N_2)$ and that $\mathcal{A}_{R'} = \alpha_{\Psi_2}(\mathcal{A}_R)$. Applying lemma 2.7 in $D(N'_1) \cap I^-(N'_1)$ we see that $\mathcal{A}_{R'} \subset \mathcal{A}_{P'_1}$ and we have $\mathcal{A}_{P_1} = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{P'_1})$. Putting this together yields the inclusion:

$$\alpha(\mathcal{A}_{P_2}) \subset \alpha(\mathcal{A}_R) = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{R'}) \subset \alpha_{\Psi_1}^{-1}(\mathcal{A}_{P'_1}) = \mathcal{A}_{P_1}.$$

Similarly we have $\mathcal{A}_{Q_1} = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{Q'_1})$, $\mathcal{A}_{Q'_2} = \alpha_{\Psi_2}(\mathcal{A}_{Q_2})$ and $\mathcal{A}_{Q'_1} \subset \mathcal{A}_{Q'_2}$ by lemma 2.7. This yields the inclusion:

$$\alpha(\mathcal{A}_{Q_2}) = \alpha_{\Psi_1}^{-1}(\mathcal{A}_{Q'_2}) \supset \alpha_{\Psi_1}^{-1}(\mathcal{A}_{Q'_1}) = \mathcal{A}_{Q_1}.$$

□

4 The Reeh-Schlieder Property in Curved Spacetime

The spacetime deformation argument of the previous section will have some consequences for the Reeh-Schlieder property that we describe in the current section. Unfortunately it is not clear that we can deform a Reeh-Schlieder state into another (full) Reeh-Schlieder state, but we do have the following more limited result:

Theorem 4.1 *Consider a locally covariant quantum field theory \mathbf{A} with state space \mathbf{S} which satisfies the time-slice axiom. Let M_i be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces and suppose that $\omega_1 \in S_{M_1}$ is a Reeh-Schlieder state. Then given any bounded cc-region $O_2 \subset M_2$ with non-empty causal complement, $O_2^\perp \neq \emptyset$, there is a $*$ -isomorphism $\alpha: \mathcal{A}_{M_2} \rightarrow \mathcal{A}_{M_1}$ such that $\omega_2 := \alpha^*(\omega_1)$ has the Reeh-Schlieder property for O_2 .*

Moreover, if the Cauchy surfaces of the M_i are non-compact and $P_2 \subset M_2$ is a bounded cc-region, then there is a bounded cc-region $Q_2 \subset P_2^\perp$ for which ω_2 has the Reeh-Schlieder property.

Proof. For the first statement let α and U_1 be as in the first part of proposition 3.4 and note that α gives rise to a unitary map $\mathbb{U}_\alpha: \mathcal{H}_{\omega_2} \rightarrow \mathcal{H}_{\omega_1}$. This map is the expression of the essential uniqueness

of the GNS-representation, so that $\mathbb{U}_\alpha \Omega_{\omega_2} = \Omega_{\omega_1}$ and $\mathbb{U}_\alpha \pi_{\omega_2} \mathbb{U}_\alpha^* = \pi_{\omega_1} \circ \alpha$. The Reeh-Schlieder property for O_2 then follows from the observation that $\mathbb{U}_\alpha \pi_{\omega_2}(\mathcal{A}_{O_2}) \mathbb{U}_\alpha^* \supset \pi_{\omega_1}(\mathcal{A}_{U_1})$:

$$\overline{\pi_{\omega_2}(\mathcal{A}_{O_2}) \Omega_{\omega_2}} \supset \overline{\mathbb{U}_\alpha^* \pi_{\omega_1}(\mathcal{A}_{U_1}) \Omega_{\omega_1}} = \mathbb{U}_\alpha^* \mathcal{H}_{\omega_1} = \mathcal{H}_{\omega_2}.$$

Similarly for the second statement, given a bounded cc-region P_2 and choosing Q_1, Q_2 as in the second statement of proposition 3.4 we see that $\mathbb{U}_\alpha \pi_{\omega_2}(\mathcal{A}_{Q_2}) \mathbb{U}_\alpha^* \supset \pi_{\omega_1}(\mathcal{A}_{Q_1})$. \square

The second part of theorem 4.1 means that ω_2 is a Reeh-Schlieder state for all cc-regions that are big enough. Indeed, if V_2 is a sufficiently small cc-region then V_2^\perp is connected (recall that we work with four-dimensional spacetimes) and therefore ω_2 has the Reeh-Schlieder property for some cc-region in V_2^\perp and hence also for V_2^\perp itself.

A useful consequence of theorem 4.1 is the following:

Corollary 4.2 *In the situation of theorem 4.1 if \mathbf{A} is causal then Ω_{ω_2} is a cyclic and separating vector for $\mathcal{R}_{O_2}^{\omega_2}$. If the Cauchy surfaces are non-compact Ω_{ω_2} is a separating vector for all $\mathcal{R}_{P_2}^{\omega_2}$ where P_2 is a bounded cc-region.*

Proof. Recall that a vector is a separating vector for a von Neumann algebra \mathcal{R} iff it is a cyclic vector for the commutant \mathcal{R}' (see [11] proposition 5.5.11.). Choosing V_1 as in the first part of proposition 3.4 we have $\mathbb{U}_\alpha \pi_{\omega_2}(\mathcal{A}_{O_2}) \mathbb{U}_\alpha^* \subset \pi_{\omega_1}(\mathcal{A}_{V_1})$ by the inclusion (4). Therefore the commutant of $\mathbb{U}_\alpha \mathcal{R}_{O_2}^{\omega_2} \mathbb{U}_\alpha^*$ contains $(\mathcal{R}_{V_1}^{\omega_1})'$. As $V_1^\perp \neq \emptyset$ this commutant contains the local algebra of some cc-region for which Ω_{ω_1} is cyclic. Hence Ω_{ω_1} is a separating vector for $\mathcal{R}_{V_1}^{\omega_1}$ and Ω_{ω_2} for $\mathcal{R}_{O_2}^{\omega_2}$.

If the Cauchy surfaces are non-compact, P_2 is a bounded region and Q_2 is as in theorem 4.1, then $(\mathcal{R}_{P_2}^{\omega_2})'$ contains $\pi_{\omega_2}(\mathcal{A}_{Q_2})$, for which Ω_{ω_2} is cyclic. It follows that Ω_2 is separating for $\mathcal{R}_{P_2}^{\omega_2}$. \square

If the state space is locally quasi-equivalent and large enough it is possible to show the existence of full Reeh-Schlieder states. The proof uses abstract existence arguments, as opposed to the proof of theorem 4.1 which is constructive, at least in principle.

Theorem 4.3 *Consider a locally covariant quantum field theory \mathbf{A} with a locally quasi-equivalent state space \mathbf{S} which is causal and satisfies the time-slice axiom. Assume that \mathbf{S} is maximal in the sense that for any state ω on some \mathcal{A}_M which is locally quasi-equivalent to a state in S_M we have $\omega \in S_M$.*

Let M_i , $i = 1, 2$, be two globally hyperbolic spacetimes with diffeomorphic non-compact Cauchy surfaces and assume that ω_1 is a Reeh-Schlieder state on M_1 . Then S_{M_2} contains a (full) Reeh-Schlieder state.

Proof. Let $\{O_n\}_{n \in \mathbb{N}}$ be a countable cover of \mathcal{M}_2 consisting of bounded cc-regions with non-empty causal complement. We then apply theorem 4.1 to each O_n to obtain a sequence of states $\omega_2^n \in S_{M_2}$ which have the Reeh-Schlieder property for O_n . We write $\omega := \omega_2^1$ and let $(\pi, \Omega, \mathcal{H})$ denote its GNS-representation.

For all $n \geq 2$ we now find a bounded cc-region $V_n \subset \mathcal{M}_2$ such that $V_n \supset O_1 \cup O_n$. For this purpose we first choose a Cauchy surface $C \subset \mathcal{M}_2$ and note that $K_n := C \cap J(\overline{O_n})$ is compact. Letting $L_n \subset C$ be a compact connected set containing $K_1 \cup K_n$ in its interior it suffices to choose $V_n := \text{int}(D(L_n)) \cap I^-(C_+) \cap I^+(C_-)$ for Cauchy surfaces C_\pm to the future resp. past of O_1 , O_n and C . Note that Ω and $\Omega_{\omega_2^n}$ are cyclic and separating vectors for $\mathcal{R}_{V_n}^\omega$ and $\mathcal{R}_{V_n}^{\omega_2^n}$ respectively by $O_1 \cup O_n \subset V_n$ and by corollary 4.2. Because ω and ω_2^n are locally quasi-equivalent there is a *-isomorphism $\phi: \mathcal{R}_{V_n}^{\omega_2^n} \rightarrow \mathcal{R}_{V_n}^\omega$. In the presence of the cyclic and separating vectors ϕ is implemented by a unitary map $\mathbb{U}_n: \mathcal{H}_{\omega_2^n} \rightarrow \mathcal{H}$ (see [11] theorem 7.2.9). We claim that $\psi_n := \mathbb{U}_n \Omega_{\omega_2^n}$ is cyclic for $\mathcal{R}_{O_n}^\omega$. Indeed, by the definition of quasi-equivalence we have $\phi \circ \pi_{\omega_2^n} = \pi_\omega$ on \mathcal{A}_{V_n} , so

$$\overline{\pi_\omega(\mathcal{A}_{O_n}) \psi_n} = \overline{\mathbb{U}_n \pi_{\omega_2^n}(\mathcal{A}_{O_n}) \Omega_{\omega_2^n}} = \mathbb{U}_n \mathcal{H}_{\omega_2^n} = \mathcal{H}_\omega.$$

We now apply the results of [7] to conclude that \mathcal{H} contains a dense set of vectors ψ which are cyclic and separating for all $\mathcal{R}_{O_n}^\omega$ simultaneously. Because each cc-region $O \subset \mathcal{M}_2$ contains

some O_n we see that $\omega_\psi : A \mapsto \frac{\langle \psi, \pi_\omega(A)\psi \rangle}{\|\psi\|^2}$ defines a full Reeh-Schlieder state. Finally, because the GNS-representation of ω_ψ is just (π, ψ, \mathcal{H}) we see that it is locally quasi-equivalent to ω and hence $\omega_\psi \in S_{M_2}$. \square

In situations of physical interest it remains to be seen whether the state space is big enough to contain such Reeh-Schlieder states. However, theorem 4.1 is already enough for some applications, such as the following conclusion concerning the type of local von Neumann algebras

Corollary 4.4 *Consider a nowhere-classical causal locally covariant quantum field theory \mathbf{A} with a locally quasi-equivalent state space \mathbf{S} which satisfy the time-slice axiom. Let M_i be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces and let $\omega_1 \in S_{M_1}$ be a Reeh-Schlieder state. Then for any state $\omega \in S_{M_i}$ and any cc-region $O \subset \mathcal{M}_i$ the local von Neumann algebra \mathcal{R}_O^ω is not finite.*

Proof. We will use proposition 5.5.3 in [1], which says that \mathcal{R}_O^ω is not finite if the GNS-vector Ω is a cyclic and separating vector for \mathcal{R}_O^ω and for a proper sub-algebra \mathcal{R}_V^ω . Note that we can drop the superscript ω if O and V are bounded, by local quasi-equivalence.

First we consider M_1 . For any bounded cc-region $O_1 \subset \mathcal{M}_1$ such that $O_1^\perp \neq \emptyset$ we can find bounded cc-regions $O' \subset O_1^\perp$ and $U, V \subset O_1$ such that $U \subset V^\perp$. By the Reeh-Schlieder property the GNS-vector Ω_{ω_1} is cyclic for \mathcal{R}_V and hence also for \mathcal{R}_{O_1} . Moreover it is cyclic for $\mathcal{R}_{O_1}' \supset \mathcal{R}_{O'}$ and therefore it is separating for \mathcal{R}_{O_1} and \mathcal{R}_V . Now suppose that $\mathcal{R}_{O_1} = \mathcal{R}_V$. Then, by causality:

$$\pi_\omega(\mathcal{A}_U) \subset \pi_\omega(\mathcal{A}_V)' = \pi_\omega(\mathcal{A}_{O_1})' \subset \pi_\omega(\mathcal{A}_U)'.$$

It follows that $\mathcal{R}_U \subset \mathcal{R}_U'$, which contradicts the nowhere classicality. Therefore, the inclusion $\mathcal{R}_V \subset \mathcal{R}_{O_1}$ must be proper and the cited theorem applies. Of course, if $O \subset \mathcal{M}_1$ is a cc-region that is not bounded, then it contains a bounded sub-cc-region O_1 as above and $\mathcal{R}_O^\omega \supset \mathcal{R}_{O_1}^\omega \simeq \mathcal{R}_{O_1}$ isn't finite either for any $\omega \in S_{M_1}$. (If V is a partial isometry in the smaller algebra such that $I = V^*V$ and $E := VV^* < I$ then the same V shows that I is not finite in the larger algebra.)

Next we consider M_2 and let $O \subset \mathcal{M}_2$ be any cc-region. It contains a cc-region O_2 with $O_2^\perp \neq \emptyset$, so we can apply theorem 4.1. Using the unitary map $\mathbf{U}_\alpha : \mathcal{H}_{\omega_2} \rightarrow \mathcal{H}_{\omega_1}$ we see that $\mathcal{R}_{O_2} \simeq \mathcal{R}_{O_2}^{\omega_2}$ contains $\alpha^{-1}(\mathcal{R}_{O_1}^{\omega_1})$, which is not finite by the first paragraph. Hence \mathcal{R}_{O_2} is not finite and the statement for O then follows again by inclusion. \square

Instead of the nowhere classicality we could have assumed that the local von Neumann algebras in M_1 are infinite, which allows us to derive the same conclusion for M_2 . Unfortunately it is in general impossible to completely derive the type of the local algebras using this kind of argument. Even if we know the types of the algebras \mathcal{A}_{U_1} and \mathcal{A}_{V_1} in the inclusions (4), we can't deduce the type of \mathcal{A}_{O_2} .

Another important consequence of proposition 4.1 is that corollary 4.2 enables us to apply the Tomita-Takesaki modular theory to $\mathcal{R}_{O_2}^{\omega_2}$ (or to the von Neumann algebra of any bounded cc-region V_2 which contains O_2 , if the Cauchy surfaces are non-compact). More precisely, let $O_2 \subset \mathcal{M}_2$ be given and let $U_1, V_1 \subset \mathcal{M}_1$ be the bounded cc-regions and $\alpha : \mathcal{M}_2 \rightarrow \mathcal{M}_1$ the *-isomorphism of proposition 3.4, so that $\mathcal{A}_{O_1} \subset \alpha(\mathcal{A}_{O_2}) \subset \mathcal{A}_{V_1}$. We can then define $\mathcal{R} := \mathbf{U}_\alpha \mathcal{R}_{O_2}^{\omega_2} \mathbf{U}_\alpha^*$ and obtain $\mathcal{R}_{U_1}^{\omega_1} \subset \mathcal{R} \subset \mathcal{R}_{V_1}^{\omega_1}$. It is then clear that the respective Tomita-operators are extensions of each other, $S_{U_1} \subset S_\mathcal{R} \subset S_{V_1}$ (see e.g. [11]).

5 The free scalar field

As an example we will consider the free scalar field, which can be quantised using the Weyl algebra (see [6]). For a globally hyperbolic spacetime M the algebra \mathcal{A}_M is defined as follows. We let $E := E^+ - E^-$ denote the difference of the advanced and retarded fundamental solution of the Klein-Gordon operator $\nabla^a \nabla_a + m^2$ for a given mass $m \geq 0$. The linear space $H := E(C_0^\infty(\mathcal{M}))$ has a non-degenerate symplectic form defined by $\sigma(Ef, Eg) := \int_{\mathcal{M}} fEg$, where we integrate with

respect to the volume element determined by the metric. To every $Ef \in H$ we can then associate an element $W(Ef)$ subject to the relations

$$W(Ef)^* = W(-Ef), \quad W(Ef)W(Eg) = e^{-\frac{i}{2}\sigma(Ef, Eg)}W(E(f+g)).$$

These elements form a $*$ -algebra that can be given a norm and completed to a C^* -algebra \mathcal{A}_M . It is shown in [4] theorem 2.2 that the scalar free field is an example of a locally covariant quantum field theory which is causal. It satisfies part of the time-slice axiom, namely if $O \subset M$ contains a Cauchy surface then $\mathcal{A}_O = \mathcal{A}_M$.¹

A state ω on \mathcal{A}_M is called regular if the group of unitary operators $\lambda \mapsto \pi_\omega(W(\lambda Ef))$ is strongly continuous for each f . It then has a self-adjoint (unbounded) generator $\Phi_\omega(f)$ and we can define the Hilbert-space valued distribution $\phi_\omega(f) := \Phi_\omega(f)\Omega_\omega$. A regular state is quasi-free iff the two-point function

$$w_2(f, h) := \langle \phi_\omega(\bar{f}), \phi_\omega(h) \rangle, \quad f, h \in C_0^\infty(\mathcal{M})$$

determines the state by $\omega(W(Ef)) = e^{-w_2(f, f)}$. A quasi-free state is Hadamard iff $WF_\infty(\phi_\omega(\cdot)) \subset \bar{V}^+$, where $\bar{V}^+ \subset T^*\mathcal{M}$ denotes the cone of future directed causal covectors of the spacetime (see [17] proposition 6.1). Quasi-free Hadamard states exist on all globally hyperbolic spacetimes (see [8]) and they are believed to be the most suitable states to play a role similar to the vacuum in Minkowski spacetime. For this reason we will want to choose a state space S_M which contains all quasi-free Hadamard states. If we choose these states only it can be shown that we get a locally quasi-equivalent state space (see [19] theorem 3.6) and the time-slice axiom is satisfied (see [14] theorem 5.1 and the subsequent discussion).

We may now apply the results of section 4:

Proposition 5.1 *Let M be a globally hyperbolic spacetime, let $O \subset \mathcal{M}$ a bounded cc-region with non-empty causal complement and assume that the mass $m > 0$ is strictly positive. Then there is a Hadamard state ω on \mathcal{A}_M which has the Reeh-Schlieder property for O . The vector Ω_ω is cyclic and separating for \mathcal{R}_O . For all bounded cc-regions $V \subset \mathcal{M}$ the local von Neumann algebra \mathcal{R}_V is not finite. Moreover, if the Cauchy surfaces of M are non-compact then Ω_ω is a separating vector for all \mathcal{R}_V .*

Proof. The theory is causal, satisfies the time-slice axiom and the state space is locally quasi-equivalent. Moreover, the theory is nowhere classical. To see this we note that the local C^* -algebras are non-commutative and simple, so the representations π_ω are faithful. Now we can find an ultrastatic (and hence stationary) spacetime M' diffeomorphic to M . Because $m > 0$ we may apply the results of [12], which imply the existence of a regular quasi-free ground state ω' on M' . This state has the Reeh-Schlieder property (see [16]) and is Hadamard because it satisfies the microlocal spectrum condition (see [17, 14]). The conclusions now follow immediately from theorem 4.1 and the corollaries 4.2 and 4.4. Note that stronger results on the type of the local algebras are known, [19]. \square

If we would enlarge our state space and allow any state that is locally quasi-equivalent to a quasi-free Hadamard state, then it follows from theorem 4.3 that it also contains full Reeh-Schlieder states. In fact, if ω is a suitable quasi-free Hadamard state on \mathcal{A}_M then the proof of theorem 4.3 shows that \mathcal{H}_ω contains a dense G_δ of vectors which define Reeh-Schlieder states. An important question is how many states are both Hadamard and Reeh-Schlieder states. As a partial answer we wish to note the following. If a vector $\psi \in \mathcal{H}_\omega$ defines a Hadamard state then it must be in the domain of the unbounded self-adjoint operator $\Phi_\omega(f)$ for some real-valued test function f . We then apply

Proposition 5.2 *The domain of an unbounded self-adjoint operator T on a Hilbert space \mathcal{H} is a meagre F_σ , (i.e. the complement of a dense G_δ).*

¹Note that this is what [4] call the time slice axiom. In our definition, however, we also need to choose a suitable state space functor so that we get isomorphisms of the sets of states too.

Proof. For each $n \in \mathbb{N}$ we define $V_n := \{\psi \in \mathcal{H} \mid \|T\psi\| \leq n\}$ and note that $\text{dom}(T) = \cup_n V_n$. The sets V_n are nowhere dense because T is unbounded. They are also closed because for a Cauchy sequence $\psi_i \rightarrow \psi$ with $\psi_i \in V_n$ we have $\|TE_{[-r,r]}\psi\| \leq \|TE_{[-r,r]}(\psi - \psi_i)\| + \|TE_{[-r,r]}\psi_i\| \leq r\|\psi - \psi_i\| + n$, where $E_{[-r,r]}$ is the spectral projection of T on the interval $[-r, r]$. Taking $i \rightarrow \infty$ shows that $\|TE_{[-r,r]}\psi\| \leq n$ for all r and hence $\|T\psi\| \leq n$, i.e. $\psi \in V_n$. This completes the proof. \square

It follows that most Reeh-Schlieder states are not Hadamard. The converse question, how many Hadamard states are Reeh-Schlieder states, remains open.

6 Conclusions

If one accepts locally covariant quantum field theory as a suitable axiomatic framework to describe quantum field theories in curved spacetime then one only needs to assume the very natural time-slice axiom in order to use the general technique of spacetime deformation. The geometrical ideas behind deformation results like proposition 3.3 are insightful, even though the proofs can become a bit involved. It should be noted, however, that these geometrical results, possibly combined with other assumptions such as causality, have immediate consequences on the algebraic side which are not hard to prove. This we have seen in section 4, where most proofs follow easily from the deformation, with the exception of theorem 4.3.

Concerning the Reeh-Schlieder property we have shown that a Reeh-Schlieder state on one spacetime can be deformed in such a way that it gives a state on a diffeomorphic spacetime which is a Reeh-Schlieder state for a given cc-region. It is even possible to get full Reeh-Schlieder states, but it is not clear whether these are “physical” enough to belong to a state space of interest. Nevertheless, our results allow us to draw conclusions about the type of local von Neumann algebras and they open up the way to use Tomita-Takesaki theory in curved spacetime.

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